Indian Statistical Institute, Bangalore

B. Math (Hons.) Second Year

Second Semester - Ordinary Differential Equations

Final Exam | Date: April 25, 2025

Maximum Marks: 50 | Duration: 3 hours

ANSWER ALL QUESTIONS

(1) (a) Discuss the existence and uniqueness of the solution of the following *initial value problem*

$$y' = y^{\frac{1}{3}} + x, \quad y(0) = y_0$$

whenever $y_0 = 0$ and $y_0 = 1$. What can you conclude about the existence and uniqueness theorem? [4]

(b) Let $x(t)$ be a continuous nonnegative function such that

$$x(t) \leq a + \int_{t_0}^{t} [b + cx(s)]ds, \quad \text{for } t \geq t_0,$$

where $a, b \geq 0$ and $c > 0$ (a, b, c are constants). Then show that for $t \geq t_0$, $x(t)$ satisfies

$$x(t) \leq \left(\frac{b}{c}\right)(\exp(c(t - t_0)) - 1) + a \exp c(t - t_0).$$

[6]

(2) (a) Let y_1 and y_2 be two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I, \tag{0.1}$$

where $I \subset \mathbb{R}$ is an interval and $p(x), q(x)$ are continuous functions on I. Show that $\varphi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions if and only if $\alpha\delta \neq \beta\gamma$. [2]

(b) Let $y_1, y_2 \in C^2(I)$. Show that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in I if and only if there exists continuous functions $p(x), q(x)$ on I such that (0.1) has y_1, y_2 as independent solution. [4]

(c) By using the *method of variation of parameters* find the general solution of

$$x^2y'' - x(x + 2)y' + (x + 2)y = x^3, \quad x > 0.$$

[Hint: $y = x$ is a solution of the homogeneous part] [4]

(3) (a) Consider $u''+q(x)u = 0$ on the interval $I := (0, \infty)$. Show that if $q(x) \geq m^2$ for all $x \in I$, any non-trivial solution $u(x)$ has infinitely many zeros, and distance between two consecutive zeros is at most π/m and if $q(x) \leq m^2$ for all $x \in I$, then $u(x)$ has infinitely many zeros and distance between two consecutive zeros is at least π/m. [5]

(b) Verify that the origin is a regular singular point and find two linearly independent solutions of $9x^2y'' + (9x^2+2)y = 0$ by using the *Frobenius method*. [5]

(4) (a) Define a *stable*, *asymptotically stable* critical point, and *Liapunov function* of an autonomous system. What are the possible behaviors, depending on γ, of the solutions to the linear system

$$\begin{cases} \frac{dx}{dt} = -\gamma x - y \\ \frac{dy}{dt} = x - \gamma y. \end{cases}$$

Also, find the critical point and discuss the stability of the critical point along with the nature of the point. [3+2]

(b) Define the *simple critical point* of nonlinear system. Consider the nonlinear system

$$\begin{cases} \frac{dx}{dt} = \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \\ \frac{dy}{dt} = x + \frac{1}{2}y - \frac{1}{2}(y^2 + yx^2). \end{cases}$$

Show that the origin is the simple critical point, and then discuss the type and stability of the critical point. [1+4]

(5) (a) Find the value of ζ for which the equation

$$(xy^2 + \zeta x^2 y)dx + (x+y)x^2 dy = 0$$

is exact and then solve it using that value of ζ.

Or, Consider the equation $y' + ay = b(x)$, where a is a constant such that real part of a is strictly positive, and $b(x)$ is a continuous function on $0 \leq x < \infty$ such that $b(x) \to \beta$ as $x \to \infty$. Prove that every solution of this equation tends to $\frac{\beta}{a}$ as $x \to \infty$. [5]

(b) Using the *Improved Euler method* to compute the approximate value $y(0.2)$ of the solution of the *initial value problem*

$$y' = 1 - x + 4y \quad y(0) = 1$$

on the interval $[0,1]$ with step size $h = 0.1$. [5]

Good luck!!